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A mathematical model of segregation patterns in residential neighbourhoods

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Abstract. A mathematical model is proposed which describes the dynamics and the spatial distributions of two population groups, where migration is driven by considerations of socioeconomic status. The model associates segregation with instabilities of spatially uniform mixed population states. These instabilities lead to a wide range of segregation forms including: (a) variable (weak) segregation where the population is everywhere mixed and the spatial variability is controlled by a 'status-gap' parameter, (b) strong segregation, where nearby neighbourhoods consists of pure (unmixed) population groups, and (c) intermediate forms involving enclaves of a pure population group in neighbourhoods of mixed population. The model associates tipping-point phenomena with the existence of an unstable mixed population state which introduces a threshold for population inversion. The model predicts that uneven invasions of one population group into another may result from interface instabilities rather than from urban heterogeneities.

1 Introduction

Residential segregation in urban areas has been studied extensively by scientists from different fields, including economics, sociology, and population geography (see among others, Clark, 1986; Lever and Paddison, 1998; Massey and Fischer, 2000; Morrill, 1995; Portugali, 2000; Taylor et al, 2000; van Kempen and Özüekren, 1998; White, 1998). The ongoing interest in this subject is the result, in part, of the common perception of residential segregation as an undesirable and even dangerous process. As commonly acknowledged, residential segregation restricts the participation of disadvantaged population groups (specifically minorities and new immigrants) in various aspects of civil society, limits socioeconomic opportunities of weak population strata, and has negative effects on various aspects of local development, such as provision of commercial facilities and social services in neighbourhoods with economically weak populations (van Kempen and Özüekren, 1998).

A popular approach to explaining intraurban segregation patterns is based on the ecological theory of the Chicago School (Park et al, 1925; see also Aldrich, 1975; Lee and Wood, 1991; Schwirian, 1983; Zang, 2000) and related theories (see, for example, Bassett and Short, 1980; Sarre et al, 1989). According to this approach, city residents differ with respect to occupation, income, and education, and these differences increase as the local economy grows. As people of similar ethnic background, income, and environmental preferences seek areas with similar social and environmental characteristics, intraurban segregation develops (Massey, 1985).

The ecological approach has been criticized by 'behaviourists' for viewing the individual as an economic entity (*Homo economicus*), and thus ignoring personal preferences and perceptions. Behavioural models emphasize household characteristics affecting such choices, like the age of the household head and the household size (Adams and Gilder, 1976; Clark, 1986; White, 1987; White and Sassler, 2000).

Recent studies of urban growth and dynamics borrowed concepts from the physical sciences such as *fractals* (Batty and Longley, 1994; Makse et al, 1995; Schweitzer and Steinbink, 1997; White and Engelen, 1993) and *self-organization* (Allen, 1997; 1999; Allen et al, 1985; Clarke et al, 1996; Haken, 1985; Portugali, 2000; Portugali et al, 1994; Straussfogel, 1991; Weidlich, 1991; 1999). More relevant to the present study is the concept of self-organization, an emergent property (Popper and Eccles, 1977) of a complex system resulting from the collective behaviour [(1)] of the constituents of the system. The constituents may be the molecules of a physical object or, in the present context, the individuals who constitute a residential neighbourhood. An important example of self-organization is *pattern formation*, a phenomenon which gives rise to highly correlated nonuniform distributions of the constituents of a system, even when the system and the forces it is subjected to are homogeneous (Cross and Hohenberg, 1993). Adopting the concept of self-organization in social geography is consistent with the view of individuals not only as economic entities, but also as social entities (*Homo socialis*) who behave in a collective manner (Sonis, 1991).

The concept of self-organization has been applied to spatial segregation by several research groups using discrete and continuum mathematical modelling. Discrete models, mostly in the form of cellular automata (Gaylord and D'Andria, 1998; Portugali, 2000; Schelling, 1978), represent the urban space as a uniform grid of cells to which 'occupation states' are assigned. The state of a given cell at a given (discrete) time is determined by a uniform set of rules applied to the states of this cell and its neighbours at the previous time step. The rules reflect individuals' behaviours that are easy to formulate, and this makes the discrete modelling approach attractive. Further developments of these models have led to the rather sophisticated 'free agent on a cellular space' models (Benenson, 1998; 1999; Portugali, 2000; Portugali and Benenson, 1995; Portugali et al, 1997) which distinguish between an infrastructure layer and a superstructure layer of individual free agents. The main disadvantage of these approaches, from our point of view, is that the models are often too complicated to analyze mathematically. They become 'black boxes' that are useful for simulating urban realities but hardly help in applying concepts of dynamical systems theory (Guckenheimer and Holmes, 1983) to the urban environment, such as attractors, instabilities, and so on (Jen, 1990).

Continuum models represent an alternative approach to modelling residential segregation. They represent space and time by continuous variables and describe averaged behaviours over scales that are large relative to a single house but small with respect to the system size. Continuum models are more amenable to mathematical analysis than discrete models and, as a result, allow precise applications of concepts of dynamical systems theory to urban phenomena. A few continuum models of neighbourhood change and intraurban segregation have been proposed (Beckmann and Puu, 1990; Gurtin, 1974; Ishikawa, 1980; O'Neill, 1981; Zhang, 1998; 1989; 1990). One disadvantage

[(1)] From a dynamical system point of view a collective behaviour is achieved when the dynamics of the system are dictated by a small set of order parameters, which all other variables follow after short transients (Haken, 1985).

of these models, however, is the absence of variable segregation, that is, an element in the model that parametrizes the segregation strength. The importance of this element arises from the fact that urban neighbourhoods are rarely totally segregated; more often they have ethnically mixed populations, and the degree of mixing varies from case to case. In a recent work Yizhaq and Meron (2002) introduced a new continuum model of residential segregation involving two population groups. In contrast to earlier models, it includes a socioeconomic status variable, in addition to the population density variables. The introduction of a status variable makes it possible to model different segregation states, spanning the whole range from weak to strong segregation.

In the present paper we extend the work of Yizhaq and Meron (2002) in several respects. In addition to indirect interactions between the population groups, mediated through a socioeconomic status variable, we include direct population interactions and examine their effects. Direct interactions, or 'identity interactions' as we shall call them, may be relevant, for instance, to segregation phenomena involving different ethnic groups. We also extend the analysis to a two-dimensional space, addressing neighbourhood-change processes. We study the merging of population-group enclaves and find an instability leading to an uneven expansion of one population group into the area occupied by another one. We study two-dimensional effects of tipping phenomena and find that the success of tipping depends on the size of the minority enclave that invades a majority-populated area. Finally, we elaborate in greater depth on the relationships between sociospatial phenomena, such as tipping and invasion, and concepts of dynamical systems theory.

We wish to stress that the proposed model is not a simulation model per se, at least in the sense in which such a model is commonly defined in urban studies (see, among others, Clarke et al, 1996; Portugali, 2000; Webster and Wu, 1999; Wu and Webster, 1998). Whereas urban simulation models attempt to imitate real-world processes (such as population change, land use, and urban expansion) by setting various empirical growth rules and constraints, the proposed modelling approach is contrastingly different. *It abstracts out key segregation mechanisms from specific urban contexts and uses mathematical equations to study the general effects these mechanisms have on segregation and neighbourhood-change phenomena.* The approach helps to eliminate factors of secondary importance in determining the causes of these phenomena, and has a considerable degree of generality as compared with ad hoc urban simulation models that are tailored to specific urban conditions (such as urban growth boundaries, the location of individual land parcels, and site-specific urban morphology). Furthermore, the model equations can be subjected to in-depth mathematical analysis, making it possible to determine basic features of segregation (for instance, the different forms of segregation caused by socioeconomic considerations versus 'identity' ones). On the other hand, although the proposed model can be used to simulate various segregation phenomena in residential neighbourhoods (invasion, succession, the creation and disappearance of ethnic enclaves), as demonstrated further in the paper, the simulation capabilities of the proposed model are rather limited, compared with urban simulation models.

The paper begins with a brief review of studies of residential segregation which are relevant to the present work (section 2). The proposed mathematical model of residential segregation is then introduced (section 3). Section 4 presents stationary homogeneous solutions of the model equations and their stability. Section 5 presents time-dependent and nonhomogeneous solutions and relates them to segregation and neighbourhood-change processes from the point of view of a dynamical system. The significance and possible extensions of this study are discussed in section 6.

2 Background studies

In an important paper dealing with the causes and consequences of residential segregation, Massey (1985) suggests that differences between urban neighbourhoods are characterized by three main variables—socioeocnomic status, family status, and ethnicity. In respect to these variables, residential segregation is caused by the interplay between two 'offsetting ecological processes': residential succession and spatial assimilation. According to his argument, residential succession is fed by chain migration (the accumulation of immigrant or migrant groups in a particular urban area) and institutionalization (the establishment of cultural and economic institutions by ethnic groups). In response to these factors, "ethnic neighbourhoods form in such a way that the extent of segregation between groups reflects the social, cultural, and economic distance between them" (Massey, 1985, page 321). The opposing force to segregation is, according to Massey, spatial assimilation: "segregation dissipates over time through a process ... driven by acculturation and socio-economic mobility" (page 321).

Poulsen et al (2002) also argue that the fragmentation of cities into ethnic enclaves results from two interrelated processes: assimilation and ghettoization. Whereas assimilation leads to reduction in segregation as the economic status of a group improves, ghettoization has the opposite effect: it enhances spatial segregation because of the economic disadvantage of a group's members or because of overt discrimination.

Housing affordability and kinship relationships may also be driving forces behind residential segregation. As new immigrants and ethnic minorities move to areas of affordable housing, and to residential areas populated by their fellow immigrants (where they can count on information and help from friends and relatives), intra-urban enclaves of ethnic neighbourhoods are formed (Owusu, 1999). Out-migration of the young and educated from low-status neighbourhoods further strengthens segregation trends (Lloyd, 1998). In some cases, residential segregation is driven by considerations of ethnic affinity, and is voluntary, rather than economically enforced. Thus, Glebe's (1986) study of Japanese migration in Dusseldorf, Germany indicates that the well-to-do Japanese avoid both German-populated neighbourhoods and neighbourhoods populated by low-status guest workers, preferring to settle in mostly Japanese-populated residential areas.

In his study of segregation patterns in King County, Morrill (1995) also argues that segregation is one of the geographic expressions of inequality, which is, most often, a result of social discrimination in the housing market, possibly politically and economically enforced. One of the most interesting observations made by Morrill is that there are to be *no* 'pure' cases of residential segregation. As his analysis shows, only 14.5% of blacks and only 7.5% of Asians resided in residential areas in which they constitute the majority of population. Concurrently, in most black areas included in the study, population tended to be ethnically heterogeneous.

This conclusion reinforces the results of an earlier survey of residential segregation in US cities, carried out by Clark (1986). According to this study, black households clearly prefer mixed neighbourhoods involving comparable portions of black and white residents. Another important conclusion of this survey is that economic status is the strongest factor affecting segregation in residential areas; this factor, although not acting alone, accounts for 30%–70% of the cases of racial separation. Similar conclusions on the overpowering effect of economic factors as driving forces behind residential segregation are mentioned by authors of other empirical studies (see, for example, Phillips, 1998; Portnov, 2002; White, 1998).

3 Proposed model

The studies described in section 2 point toward two conclusions. The first is that segregation is affected by counteracting forces—residential succession and spatial assimilation (Massey, 1985)—and therefore is variable; it can be strong, involving enclaves of pure populations, but very often it assumes weaker, mixed populations forms (Morrill, 1995). The second conclusion is that socioeconomic status is one of the major considerations affecting segregation (Clark, 1986). In modelling residential segregation we take these conclusions into account by introducing a socioeconomic status variable.

For simplicity we restrict ourselves to segregation patterns involving two population groups only. The approach we pursue here, however, can be used to include a third population group as well. This will make the model more complex but still tractable. The two groups we consider in the present model are characterized by different socioeconomic status, which may reflect gaps in education, occupation, income, etc. The two population groups may thus describe natives and immigrants, natives and guest workers, two different ethnic groups, and so on.

We further assume that geographically the residential area under consideration is sufficiently small and may be considered homogeneous in respect of infrastructure development, types of housing, accessibility to the city centre, and other foci of population attraction. This assumption can be relaxed by introducing space-dependent parameters.

3.1 Equations for population densities

The two population groups are characterized by their densities $u(\boldsymbol{x}, t)$ and $v(\boldsymbol{x}, \mathbf{t})$, where $\boldsymbol{x} = (x, y)$ represents the spatial coordinates and t is time. The socioeconomic status at location $\boldsymbol{x}$ and time t is denoted by $s(\boldsymbol{x}, t)$. We associate the u-residents with high socioeconomic status and the v-residents with low status. The changes of the two densities in time consist of growth contributions, G_u and G_v, and migration contributions, M_u and M_v. The principal difference between these two components is as follows. The contributions G_u and G_v change the overall population size of the system (causing either growth or decline in the number of residents). Concurrently, the contributions M_u and M_v reflect the internal circulation of population within the system; these contributions do not change the overall population size of the system.

Rate of change:		**Growth contributions:**		**Migration contributions:**
Change of local population density per unit of time	=	*Natural growth*—exponential growth constrained by limited space *Growth due to migration*—migration into and out of the system, driven by status considerations.	+	*Density-driven migration*—migration within the system to less populated areas *Status-driven migration*—migration within the system following status differences between populations.

$$\underbrace{\frac{\partial u}{\partial t} \text{ or } \frac{\partial v}{\partial t}} \qquad \underbrace{G_u \text{ or } G_v} \qquad \underbrace{M_u \text{ or } M_v} \qquad (1)$$

Schematically, the contributions in question can be illustrated by the following equation: the growth terms are modelled by

$$G_u(u, s) = (\alpha_1 - \alpha_2 u)u + \alpha_3(s - s_\mathrm{R})u, \qquad G_v(v, s) = (\beta_1 - \beta_2 v)v - \beta_3(s - s_\mathrm{R})v, \quad (2)$$

where all coefficients, α_i and β_i $(i = 1, 2, 3)$ are positive. The contributions $(\alpha_1 - \alpha_2 u)u$ and $(\beta_1 - \beta_2 v)v$ describe *logistic* growth. Unrestricted natural growth (described by $\alpha_1 u$

and $\beta_1 v$) leads to an exponential increase. This growth, however, may slow down and saturate subsequently ($-\alpha_2 u^2$ and $-\beta_2 v^2$) because of limited local housing capacity. The contributions $\alpha_3(s - s_R)u$ and $-\beta_3(s - s_R)v$, where s_R is a constant reference status, describe changes in local population densities through 'status-driven' migration. According to these terms, a neighbourhood with a high status ($s > s_R$) attracts high-income (u-residents) and drives out low-income residents (v-residents). This 'repulsion' effect may be caused by high rental prices or high property taxes, specifically when the ad valorem (for example, value-based) system of property taxation is in effect. Because low-income residents who already own properties in high-status neighbourhoods cannot always afford to pay high property taxes, they may consider selling their properties and moving elsewhere. Concurrently, a low-status neighbourhood ($s < s_R$) attracts v-residents and drives out u-residents.

We chose these contributions to be proportional to the densities u and v in order to account for the effects of *institutionalization* (Massey, 1985; van Kempen and Özüekren, 1998). This effect can be explained as follows: a higher concentration of v-residents in a low-status neighbourhood strengthens the local network of social institutions (ethnic shops, clubs, etc), which facilitate the absorption of new v-residents.

The migration terms have two types of contributions:

(a) Migration from densely populated regions to less populated ones is modelled by

$$\int [u(\boldsymbol{x}') - u(\boldsymbol{x})]\, p_u(\boldsymbol{x}' - \boldsymbol{x}) \mathrm{d}\boldsymbol{x}' \ ,$$

for temporal change of u, and by

$$\int [v(\boldsymbol{x}') - v(\boldsymbol{x})] q_v(\boldsymbol{x}' - \boldsymbol{x}) \mathrm{d}\boldsymbol{x}' \ ,$$

for the temporal change of v.

(b) Migration driven by status considerations is modelled by

$$-\int [s(\boldsymbol{x}') - s(\boldsymbol{x})]\, p_s(\boldsymbol{x}' - \boldsymbol{x}) \mathrm{d}\boldsymbol{x}' \ ,$$

for the temporal change of u, and by

$$\int [s(\boldsymbol{x}') - s(\boldsymbol{x})] q_s(\boldsymbol{x}' - \boldsymbol{x}) \mathrm{d}\boldsymbol{x}' \ ,$$

for the temporal change of v. In these expressions, $\mathrm{d}\boldsymbol{x} = \mathrm{d}x'\mathrm{d}y'$ is an infinitesimal area element representing a small part of the residential area that is still large in comparison with a single house, p_u, p_s, q_v, q_s are positive weight functions, and the integration extends over the whole system.

The integral forms of these terms reflect the fact that intraurban migration is often nonlocal: a migrant may move to a relatively distant neighbourhood if this neighbourhood meets his or her needs. Thus, the density of u-residents at location $\boldsymbol{x}$ will increase thanks to migration from densely populated places $\boldsymbol{x}'$ [$u(\boldsymbol{x}') - u(\boldsymbol{x}) > 0$] and decrease thanks to migration to less populated areas [$u(\boldsymbol{x}') - u(\boldsymbol{x}) < 0$]. Similarly, the density of u-residents at location $\boldsymbol{x}$ will increase thanks to migration from places $\boldsymbol{x}'$ with lower status [$s(\boldsymbol{x}') - s(\boldsymbol{x}) < 0$], and decrease thanks to migration to places $\boldsymbol{x}'$ with higher status [$s(\boldsymbol{x}') - s(\boldsymbol{x}) > 0$]. Similar considerations hold true for v-residents except that status gradients act in opposite sense.

There are also considerations that favour *local migration*, that is, migration within adjacent residential blocks. Families, for example, may prefer to move within their neighbourhoods in order to avoid pulling children out of local schools or in order to retain networks of social and organizational relations. The actual decisions of where to move are also governed to a large extent by the information individuals possess about the urban area, and this information is more readily available for the areas in which individuals already live (Clarke, 1986; White, 1998).

These considerations determine the shapes of the weight functions—p_u, p_s, q_v, q_s. We assume that each of these functions has two contributions: a constant term representing nonlocal migration, where all migration distances are conditionally given equal weights, and a term which peaks at $\boldsymbol{x} = (0, 0)$ representing local migration. A possible choice for p_u is

$$p_u(\boldsymbol{x}) = a_u + b_u \exp\left(-\frac{r^2}{d^2}\right) ,$$

where $r^2 = x^2 + y^2$. In this equation, a_u is the constant contribution term, and $b_u \exp(-r^2/d^2)$ is the term that peaks at $\boldsymbol{x} = (0, 0)$ and decays as we go away from the origin (the parameter b_u represents the size of the local migration and the parameter d its spatial extent). The meaning of this term is that most migrants move over short distances (Clark, 1980). This behaviour is more typical of poor migrants (Lynn and McGeary, 1990, page 91). Similar forms can be chosen for p_s, q_v, and q_s. Expanding $u(\boldsymbol{x}')$ in a Taylor series about $u(\boldsymbol{x})$ we find

$$\int [u(\boldsymbol{x}') - u(\boldsymbol{x})]\, p_u(\boldsymbol{x} - \boldsymbol{x}')\mathrm{d}\boldsymbol{x}' \cong \alpha_4[\langle u \rangle - u(\boldsymbol{x})] + D_1 \nabla^2 u ,$$

where α_4 and D_1 are constant parameters, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian operator in the plane, and

$$\langle u \rangle = \frac{1}{A} \int_A u(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}'$$

is the average of u over the system area, A. Similar forms are obtained for the other integrals. Summing up all migration contributions, M_u for u-residents and M_v for v-residents, we obtain:

$$\begin{aligned} M_u(u, s) &= D_1 \nabla^2 u - D_2 \nabla^2 s + \alpha_4[\langle u \rangle - u(\boldsymbol{x})] - \alpha_5[\langle s \rangle - s(\boldsymbol{x})] , \\ M_v(v, s) &= D_3 \nabla^2 u + D_4 \nabla^2 s + \beta_4[\langle u \rangle - u(\boldsymbol{x})] + \beta_5[\langle s \rangle - s(\boldsymbol{x})] . \end{aligned} \tag{3}$$

The diffusion terms in equations (3) describe local migration while the other terms in equations (3) describe the net effect of nonlocal migration. The schematic form (1) can now be written in mathematical terms as

$$\frac{\partial u}{\partial t} = G_u(u, s) + M_u(u, s), \qquad \frac{\partial v}{\partial t} = G_v(v, s) + M_v(v, s) , \tag{4}$$

where G_u and G_v are given by equations (2) and M_u and M_v are given by equations (3).

3.2 Evolution equation for the status variable

So far we have discussed how socioeconomic status affects population densities. The relationship between these variables is, however, reciprocal, that is, population densities may affect the status of a neighbourhood as well: new u-residents raise the overall socioeconomic status of a neighbourhood, whereas new v-residents lower it (Galster, 1987). This is a positive feedback effect; high (low) status attracts u-residents (v-residents), which raise (lower) the status further up (down). In practice, the socioeconomic status and gradients thereof can neither grow nor decline indefinitely. For instance, the upper percentile of society may include people with different levels of wealth. To saturate the status growth (in absolute value), balancing terms should be introduced. Schematically, we may write the equation for the temporal status change as:

$$\underbrace{\begin{matrix}\textit{Change of}\\ \textit{local status}\\ \textit{in time}\end{matrix}}_{\frac{\partial s}{\partial t}} = \underbrace{\begin{matrix}\textit{Positive feedback:}\\ u\text{-residents raise the status}\\ \text{while } v\text{-residents lower it.}\end{matrix}}_{P} + \underbrace{\begin{matrix}\textit{Balance of}\\ \textit{positive}\\ \textit{feedback}\end{matrix}}_{B} \tag{5}$$

The positive feedback and balance contributions are modelled by:

$$P = \gamma_1 u - \gamma_2 v\,, \qquad B = -\gamma_3(s - s_0) - \gamma_4(s - s_0)^3 + D_5\nabla^2 s\ , \tag{6}$$

where the parameters γ_1, γ_2, γ_3, γ_4 and D_5 in equations (6) are all positive. The diffusion term in equations (6) acts to smooth out sharp status gradients, an effect that can be understood in the following way. High-status neighbourhoods generate a demand for goods and services which low-status residents in nearby neighbourhoods can provide. This acts to raise the socioeconomic status of the latter (van Kempen and van Weesep, 1998). The nonlinear term has a cubic rather than quadratic form in order to allow saturation of both growth and decline of status. As we shall see in section 4, the parameter γ_3 plays a crucial role in capturing the phenomenon of segregation and can be related to the tension between the two population groups. The schematic from (5) can now be written mathematically as

$$\frac{\partial s}{\partial t} = P(u, v) + B(s)\ , \tag{7}$$

where P and B are given by equations (6). Equations (4) and (7) constitute the model equations.

3.3 The full model and its symmetric form

Some of the parameters appearing in these equations can be eliminated by rescaling time, space, and the dynamical variables u, v, and s (Murray, 1989; Yizhaq, 2003).(2) In terms of the rescaled quantities the model equations read

$$\frac{\partial u}{\partial t} = u - u^2 + su + \nabla^2 u - \delta_1\nabla^2 s + U_{\rm NL}\ , \tag{8}$$

$$\frac{\partial v}{\partial t} = \alpha v - v^2 - \beta s v + \delta_2\nabla^2 v + \delta_3\nabla^2 s + V_{\rm NL}\ , \tag{9}$$

$$\frac{\partial s}{\partial t} = \varepsilon(u - \gamma v - \mu s) - \xi s^3 + \delta_4\nabla^2 s\ , \tag{10}$$

where

$$U_{\rm NL} = \rho_1[\langle u\rangle - u(x)] - \rho_2[\langle s\rangle - s(x)]\,, \qquad V_{\rm NL} = \sigma_1[\langle v\rangle - v(x) + \sigma_2[\langle s\rangle - s(x)].$$

To help identify solutions and instabilities of equations (8)–(10) we will often consider a symmetric local form of these equations by setting $\alpha = \beta = \delta_2 = 1$, $\delta_3 = \delta_1 = \delta$, and dropping the nonlocal terms $U_{\rm NL}$ and $V_{\rm NL}$. The model then reads

$$\frac{\partial u}{\partial t} = u - u^2 + su + \nabla^2 u - \delta\nabla^2 s\ , \tag{11}$$

(2) For example, time is rescaled by the growth rate, α_1, so that the new nondimensional time is $\alpha_1 t$. There is no loss of information in this process; the dimensional quantities (variables and parameters) can be recovered at any time by simple algebraic operations.

and

$$\frac{\partial v}{\partial t} = v - v^2 - sv + \nabla^2 v + \delta \nabla^2 s \ , \tag{12}$$

$$\frac{\partial s}{\partial t} = \varepsilon(u - v - \mu s) - \xi s^3 + \delta_4 \nabla^2 s \ . \tag{13}$$

Equations (11)–(13) are invariant under the symmetry transformation

$$u \rightarrow v, \qquad v \rightarrow u, \qquad s \rightarrow -s \ . \tag{14}$$

The significance of this symmetry is that if $u = u_0$, $v = v_0$, $s = s_0$ is a solution of equations (11)–(13) then $u = v_0$, $v = u_0$, $s = -s_0$ is also a solution. Thus, any solution for which $u \neq v$ is accompanied by a symmetric counterpart.

In the model equations (8)–(10) or (11)–(13) the interaction between the two population groups is determined by considerations of socioeconomic status. For instance, a population group may not respond immediately to population changes in its neighbourhood. It may rather observe ongoing changes in the socioeconomic status of its neighbourhood before making a decision either to move out or to stay. Mathematically, the coupling between the population densities u and v is mediated by the status variable s. We refer to this type of interaction as *indirect*.

Under certain circumstances the interactions may also be *direct*. For instance, residents of a given neighbourhood may decide to move out simply because another ethnic group moves in. These 'identity' interactions can be modelled by adding terms of the form uv to the right-hand sides of equations (11) and (12). The modified equations then read

$$\frac{\partial u}{\partial t} = u - u^2 - \zeta uv + su + \nabla^2 u - \delta \nabla^2 s \ , \tag{11$'$}$$

$$\frac{\partial v}{\partial t} = v - v^2 - \zeta uv - sv + \nabla^2 v + \delta \nabla^2 s \ . \tag{12$'$}$$

Assuming ζ is positive,(3) an increase in the density of v-residents at a given location leads to a decrease in the density of u-residents at that location and vice versa. The direct coupling terms model repulsive interactions and reflect mutual avoidance between the two groups.

In the following sections we study the model equations analytically and numerically. In integrating the model equations numerically, we use standard methods as described in (Yizhaq, 2003). The parameter values used in the numerical simulations are specified in the figure captions.

4 Uniform population states and their stability

The simplest solutions of the model equations (11)–(13) are stationary uniform solutions. The population states they represent (table 1, over) may not be encountered in urban realities, but they may provide important information for analyzing possible nonuniform and nonstationary population states. To find stationary uniform solutions we set all time and space derivatives in the model equations to zero.

First, let us consider the symmetric model (11)–(13). Solving the resulting equations, we obtain the population states shown in table 1. Note that the nonlocal terms U_{NL} and V_{NL} vanish for uniform solutions. Thus, the solutions displayed in table 1 also

(3) ζ is often referred to as a *competition coefficient* (Brown and Rothery, 1993, page 379).

Table 1. Stationary uniform solutions of the symmetric model (11) – (13).

State	Symbol	Solution $(u,\ v,\ s)$
No population	0	$(0, 0, 0)$
Symmetric mixed population	M	$(1, 1, 0)$
Nonsymmetric mixed population	N_-	$(1-\eta,\ 1+\eta,\ -\eta)$
	N_+	$(1+\eta,\ 1-\eta,\ \eta)$
Pure population	P_+	$(p,\ 0,\ s_p)$
	P_-	$(0,\ p,\ -s_p)$

solve the nonlocal symmetric model [equations (11) – (13) with the terms U_{NL} and V_{NL} included]. The parameter η in table 1 is given by

$$\eta = \left[\frac{\varepsilon(2-\mu)}{\xi}\right]^{1/2} . \tag{15}$$

The analytical expressions for ρ and s_p are fairly complicated and we prefer graphic displays such as that in figure 1(a).

We are concerned mainly with stable population states. Unstable states evolve to other (stable) states under the action of arbitrarily small perturbations and cannot describe the system in the long run. We first consider the linear stability of the states to uniform perturbations. The no-population state 0 exists for all parameters but is always unstable. This reflects our assumption that the area under consideration has all the infrastructure and facilities to serve as a residential area and therefore residents immediately occupy vacant housing units.

The range of existence and stability of all other states are summarized in the bifurcation diagram presented in figure 1(a). The diagram shows the u-component of the various solutions as functions of the parameter μ. Solid (dashed) lines represent stable (unstable) solutions or states. The parameter μ measures (in an inverse sense) the status gap that develops between distinct population states (for example, N_- and N_+)

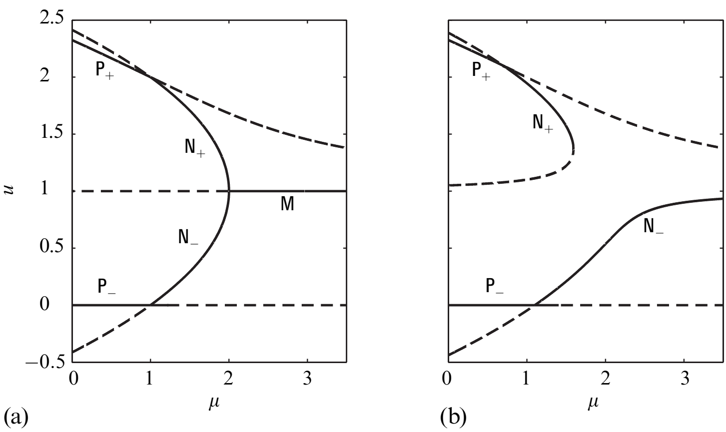

Figure 1. Bifurcation diagrams for stationary uniform solutions, showing the density of u-residents as a function of the parameter μ which measures in an inverse sense the social inequality. Solid (dashed) lines stand for stable (unstable) solutions. The symmetric model [equations (14) – (16) with $\varepsilon = 1$ and $\xi = 1$] gives rise to a pitchfork bifurcation (a) which unfolds into an imperfect pitchfork bifurcation (b) when the asymmetric model is considered [equations (11) – (13) with $\varepsilon = 1$, $\xi = 1$, $\rho_1 = \sigma_1 = \rho_2 = \sigma_2$]. Similar diagrams can be drawn for v and s. For more details see the text.

and should be regarded as a characteristic property of the two populations. According to equation (13), large values of μ imply strong damping of status deviations from the reference zero value (or s_R in the unscaled equations) and, consequently, small status gaps. Indeed according to table 1, the status gap, Δs, between N_+ and N_- is 2η, and the smaller μ the larger Δs or the socioeconomic inequality between the two populations. In the next section we will relate μ to the index of dissimilarity.

As μ is decreased below $\mu_p = 2$ the symmetric mixed population state M loses stability and the pair of asymmetric mixed population states, N_- and N_+ appear. This is a *pitchfork bifurcation*, which breaks the symmetry of the state M. The appearance of *two* new states is a consequence of the symmetry (14) of the symmetric model. The pitchfork bifurcation and the pair of asymmetric mixed population states are missing in earlier models. Upon decreasing μ further another critical value $\mu_t = 2 - \xi/\varepsilon$ appears where the asymmetric mixed states lose stability in a pair of *transcritical bifurcations*. At this critical point the two pure states P_+ and P_- gain stability.

The various states described above pertain to a symmetric system with respect to the two population groups, where characteristic properties, such as natural growth rate and mobility are equal for the two groups. In reality, people of lower socioeconomic status live in denser residential areas, their natural growth rate is often higher, and their mobility is lower (Clark, 1980; Lynn and McGeary, 1990). This asymmetry is included in the system (8)–(10). The significance of the symmetric model lies in the bifurcation structure it unravels. The behaviours of the asymmetric system (8)–(10) near the bifurcation points are captured by universal unfolding of this bifurcation structure (Golubitsky et al, 1988). Figure 1(b) shows how the pitchfork bifurcation of figure 1(a) is modified when we consider the asymmetric equations (8)–(10). It shows in particular that in the asymmetric case both the N_+ state and P_+ can coexist in contrast to the symmetric case.

Including direct (or 'identity') interaction terms, $\pm\zeta uv$ [see equations (11′) and (12′)], does not change the qualitative bifurcation picture. The main effect of the repulsive direction interaction, as figure 2 demonstrates, is to increase the status-gap range (or the range of μ) at which the pure states P_+ and P_- are stable. The pitchfork

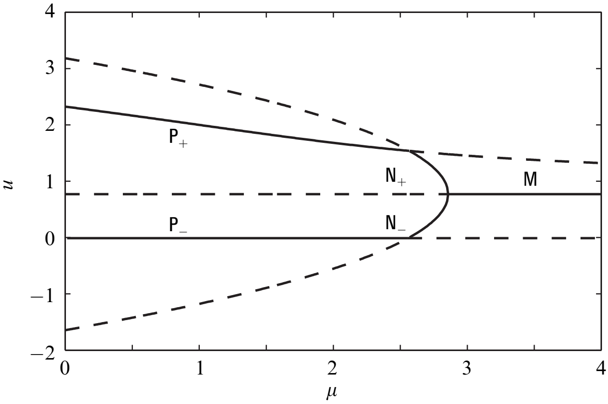

Figure 2. Bifurcation diagram for the extended symmetric model, equations (14′), (15′), and (16), with repulsive direct coupling between u and v ($\zeta = 0.3$), representing 'identity' repulsion. Solid (dashed) lines stand for stable (unstable) uniform solutions. The repulsive interaction between the two population groups extends the range of μ where the two pure states, P_+ and P_-, stably coexist. A comparison with figure 1(a) indicates that status-gap values that give rise to stable mixed population states (N_+ and N_-) in the absence of repulsive population interactions may no longer allow for stable mixed populations when these interactions exist. Parameters: $\varepsilon = 1$, $\xi = 1$.

bifurcation point shifts to higher μ values according to $\mu_p = 2/(1-\zeta)$, assuming $\zeta < 1$, while the values of the pure states are independent of ζ. As a result the intersection points of the mixed population branches, N_+ and N_-, and the pure population branches, P_+ and P_-, shift to higher μ values. The implication of this shift on segregation is that the stronger the repulsive identity interactions the stronger the segregation. Because the qualitative structure of the bifurcation diagram is not changed by the direct interaction terms we will assume in the following that $\zeta = 0$ unless we specifically consider other values.

The stationary uniform states described in table 1 may lose stability to *nonuniform* perturbations as well. Indeed, both the M state and the N_- and N_+ states may undergo Turing instabilities provided status-driven migration (parametrized by δ) is strong enough (Yizhaq, 2003). In the following we consider parameter ranges where Turing instabilities do not appear.

5 Relating model solutions to sociospatial phenomena

The analysis of the previous section reveals parameter ranges where two stable uniform solutions coexist. In the symmetric model, the N_- and N_+ states coexist in the range $\mu_t < \mu < \mu_p$, and the P_+ and P_- states coexist in the range $\mu < \mu_t$. In the asymmetric model additional forms of coexistence are possible. The coexistence of stable uniform solutions allows for nonuniform solutions pertaining to islands of one population state in a neighbourhood of the other. A variety of sociospatial structures and phenomena can be related to solutions of this kind: segregation patterns, transition zones, neighbourhood-change processes, etc. In the following subsections we present numerical solutions of the symmetric and asymmetric models describing such structures and phenomena. In solving the model equations we use no-flux boundary conditions. This excludes local migration across the boundaries of the system, but does not exclude interurban migration modelled by equation (2).

5.1 Segregation forms

The phenomenon of segregation refers to nonuniform distributions of population groups where some areas show an overrepresentation and other areas an underrepresentation of a given population group (Morrill, 1995). The bifurcation diagram for the symmetric model shown in figure 1(a) suggests three different segregation behaviours as the parameter μ is decreased, or the socioeconomic inequality is increased: no segregation ($\mu > \mu_p$), variable (weak) segregation ($\mu_t < \mu < \mu_p$), and strong segregation ($\mu < \mu_t$).

The absence of segregation for $\mu > \mu_p$ is implied by the existence of a *single* stable uniform state, the M state, representing a uniform mixed population. In the range $\mu_t < \mu < \mu_p$ [$\mu_t = 1$ in figure 1(a)], two stable uniform states coexist, the N_- and N_+ mixed population states, and segregation patterns involving domains of N_- in neighbourhoods of N_+ and vice versa are possible. The segregation is weak close to $\mu = \mu_p = 2$ because the inequality between the two asymmetric states N_- and N_+ is small, but as μ is decreased toward $\mu = \mu_t = 1$ the segregation becomes stronger. In an N_+ domain the majority of the population is affluent u-residents but a minority of high-status v-residents is able to share the same site. The opposite holds for an N_- domain. These patterns of segregation are consistent with those reported in various empirical studies (see, among others, Friedrichs, 1998; Guest and Weed, 1976; St. John and Clymer, 2000). Below $\mu = \mu_t$, the N_- and N_+ states become unstable while the pure population states, P_+ and P_-, become stable. Segregation patterns involving domains of these states describe enclaves of pure v-population in neighbourhoods of pure u-population and vice versa (Yizhaq and Meron, 2002).

A common quantitative measure of segregation is the index of dissimilarity (Massey and Denton, 1988), defined as

$$\text{Diss} = \frac{1}{2}\sum_{i=1}^{n}\left|\frac{m_{vi}}{M_v} - \frac{m_{ui}}{M_u}\right| , \tag{16}$$

where m_{vi} is the number of v-residents in the ith neighbourhood, M_v is the total number of v-residents in the whole residential area, and similar definitions hold for the u-population. The index of dissimilarity spans the range $0 \leqslant \text{Diss} \leqslant 1$; if the two populations are distributed evenly throughout the residential area, Diss $= 0$, and if the two groups reside in completely different areas, Diss $= 1$. In the context of the symmetric model, Diss $= 0$ (no segregation) for $\mu \geqslant \mu_p$ where the uniform mixed state M is stable and Diss $= 1$ (stronger segregation) for $\mu \leqslant \mu_t$ where the pure population states P_+ becomes stable.

The expression, $N_\pm = (u_\pm, v_\pm, s_\pm) = (1 \pm \eta, 1 \mp \eta, \pm \eta)$ with $\eta = [\varepsilon(2-\mu)/\xi]^{1/2}$, for the mixed population states of the symmetric model can be used to calculate the dissimilarity index in the intermediate range $\mu_\mathrm{t} < \mu < \mu_\mathrm{p}$, where the segregation is variable. For simplicity we assume that the total number of v-residents in the whole residential area is equal to the total number of u-residents, that is, $M_v = M_u \equiv M$. The number of v-residents (u-residents) in the ith neighbourhood is $m_{vi} = v_i A_i$ ($m_{ui} = u_i A_i$), where A_i is the area of the ith neighbourhood, and if the ith neighbourhood is occupied by the $\mathsf{N}_\pm$ states, $m_{vi} = v_\pm A_i$, and $m_{ui} = u_\pm A_i$. Using the definition (16) of the dissimilarity index, a straightforward calculation leads to Diss $= A\eta/M$, where A is the total residential area. The average population density in any neighbourhood is $(v_\pm + u_\pm)/2 = 1$. This is also the average density in the whole residential area, $M/A = 1$, and therefore

$$\text{Diss} = \eta = \left[\frac{\varepsilon(2-\mu)}{\xi}\right]^{1/2} . \tag{17}$$

Equation (17) describes the dependence of the index of dissimilarity on the status-gap parameter μ. At the pitchfork bifurcation ($\mu = \mu_\mathrm{p}$) where $\mathsf{N}_+ = \mathsf{N}_- = \mathsf{M}$ we should obtain Diss $(\mu_\mathrm{p}) = 0$ as the pair of asymmetric states N_+ and N_- coincide to form a single stable state M. Inserting the numerical value of μ_p ($\mu_\mathrm{p} = 2$) in equation (17) leads indeed to Diss $(\mu_\mathrm{p}) = 0$. Likewise, at the pair of transcritical bifurcations ($\mu = \mu_\mathrm{t}$), where $\mathsf{N}_\pm = \mathsf{P}_\pm$ we should obtain Diss $(\mu_\mathrm{t}) = 1$. Inserting $\mu_\mathrm{t} = 2 - \xi/\varepsilon$ in equation (17) indeed gives Diss $= 1$.

The calculation of the index of dissimilarity can easily be extended to the case of direct (ethnic) interactions ($\zeta \neq 0$). Figure 3 (over) shows graphs of Diss versus μ for increasing values of the identity-interaction parameter ζ. The graphs show two trends as ζ increases: (a) segregation begins at larger μ values or at smaller status gaps, (b) the status-gap range pertaining to variable segregation ($0 < \text{Diss} < 1$) diminishes.

The more realistic asymmetric model suggests richer behaviour. The bifurcation diagram in figure 1(b), shows for example, a parameter range where the pure population state P_- stably coexists with the mixed population state N_+. In this parameter range a segregation pattern involving enclaves of pure v-population in a neighbourhood of mixed population with a majority of u-residents, is possible, as figure 4 (over) demonstrates.

5.2 Transition zones

The interfaces between adjacent neighbourhoods form transition zones. The interfaces in figure 4, for example, show a gradual transition from a pure v-population (P_-) to a mixed population with a majority of u-residents (N_+). Figure 5 (over) shows a transition zone between a pure u-population (P_+) and a pure v-population (P_-). The width of the transition zone is affected by the parameters δ_1, δ_2, and δ_3 which represent the mobilities of u-residents and v-residents. When the pure population states are not symmetric as in figure 5, one population group (P_-) invades the other (P_+). The interfaces in figures 4

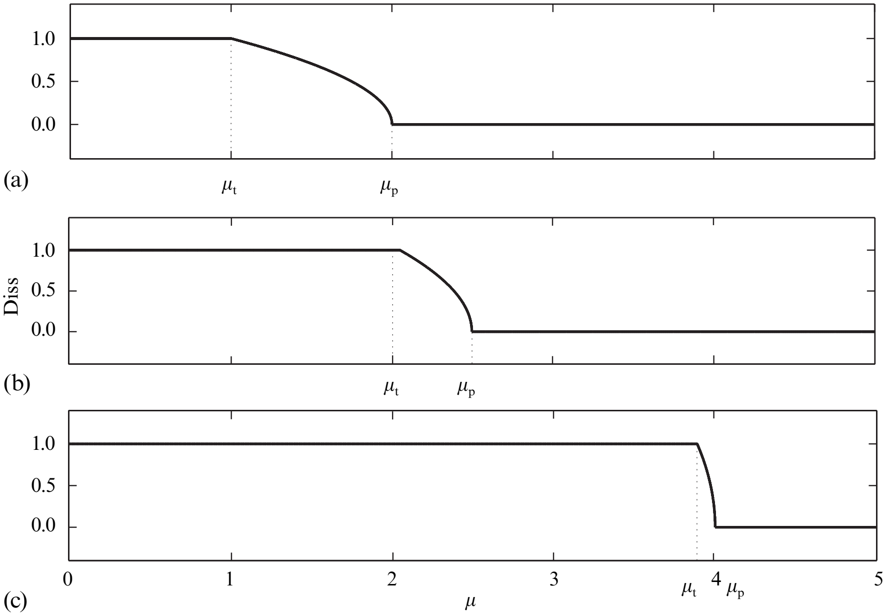

Figure 3. The effects of status gap and 'identity' repulsion on segregation. Shown are graphs of the index of dissimilarity versus the status-gap parameter, μ, at increasing direct interaction values: (a) $\zeta = 1$, (b) $\zeta = 0.2$, (c) $\zeta = 0.5$. For given ζ, segregation (Diss > 0) begins as μ is decreased below the pitchfork bifurcation threshold μ_p [see figure 1(a)]. The segregation gradually increases and becomes strong (Diss $= 1$) as μ is further decreased below the transcritical bifurcations at μ_t. As ζ increases the onset of segregation, μ_p, increases (segregation begins at lower status gaps), the onset of strong segregation, μ_t, increases as well, as the status gap range, $\mu_t < \mu < \mu_p$, pertaining to variable segregation ($0 <$ Diss < 1) diminishes. Other parameters: $\varepsilon = 1$, $\xi = 1$.

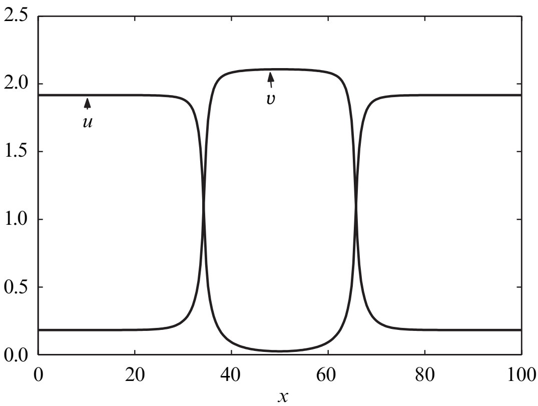

Figure 4. An enclave of pure v-population, P_-, in a neighbourhood of mixed population with a majority of u-residents, N_+, obtained with the asymmetric model [equations (11)–(13)]. Parameters: $\mu = 1.05$, $\varepsilon = 1$, $\xi = 1$, $\alpha = 1.1$, $\beta = 1$, $\gamma = 1$, $\delta_2 = 1$, $\delta_1 = \delta_3 = 0.5$, $\delta_4 = 0.05$, $\sigma_1 = \sigma_2 = \rho_1 = \rho_2 = 0$.

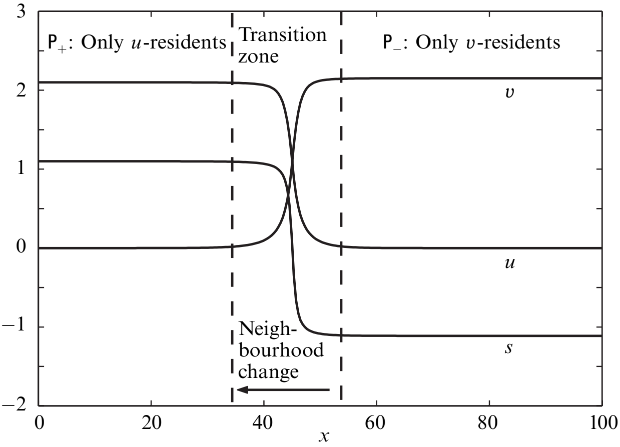

Figure 5. A transition zone (interface) between the two pure population states P_- and P_+ for the asymmetric model [equations (11)–(13)]. u-residents (v-residents) in the transitions zone have lower (higher) status values compared with the u-residents (v-residents) on the left (right) of the transition zone. A growth rate of v-residents higher than that of the u-residents ($\alpha = 1.04$) leads to invasion of the P_- state into the P_+ state. Other parameters: $\mu = 0.7$, $\varepsilon = 1$, $\xi = 1$, $\alpha = 1.04$, $\beta = 1$, $\gamma = 1$, $\delta_2 = 1$, $\delta_1 = \delta_2 = 0.2$, $\delta_4 = 0.1$, $\sigma_1 = \sigma_2 = \rho_1 = \rho_2 = 0$.

and 5 appear to be similar in shape to those found in various empirical studies (see, for example, O'Neill, 1981).

Figure 6 (over) shows in a qualitative way the dependence of the width of the transition zone on the parameter μ, which parameterizes the socioeconomic status gap, and on δ, which parameterizes population mobility. Increasing the socioeconomic inequality (or status gap) by decreasing the parameter μ leads to a narrowing of the transition zone. This observation can be interpreted as follows. The increased status gap between individuals in the transition zones creates cognitive tensions (Festinger, 1957; Portugali et al, 1997) which drive residents out of these zones, thus narrowing them. The same effect is obtained by increasing the parameter δ (mobility) for a given μ (status gap). Increasing mobility (δ) facilitates migration motivated by status considerations; residents may move out of the transition zone even when the status gap remains unchanged.

5.3 Invasion and tipping

We use the model to study two types of neighbourhood-change process: (a) invasion, involving displacements of transition zones, and (b) tipping, involving sudden reversals of population distributions once critical population thresholds have been exceeded. Invasion processes are studied by analyzing interface (or front) solutions of the model equations. Tipping processes are studied by analyzing homogeneous and 'nucleus' solutions.

5.3.1 *Invasion process*

Numerical studies of the symmetric model [equations (11)–(13)] indicate that interface solutions separating the two mixed population states, N_- and N_+, or the two pure population states, P_- and P_+, are stationary.[4] Interface solutions of the more generic asymmetric model [equations (8)–(10)], however, may propagate. Transition-zone

[4] A symmetry between the two populations states does not rule out interfaces moving at constant speeds. Such moving interfaces may arise in pitchfork front bifurcations (Coullet et al, 1990; Hagberg and Meron, 1994; Meron, 1999). However, no indications for the existence of such a bifurcation have been found in the present model.

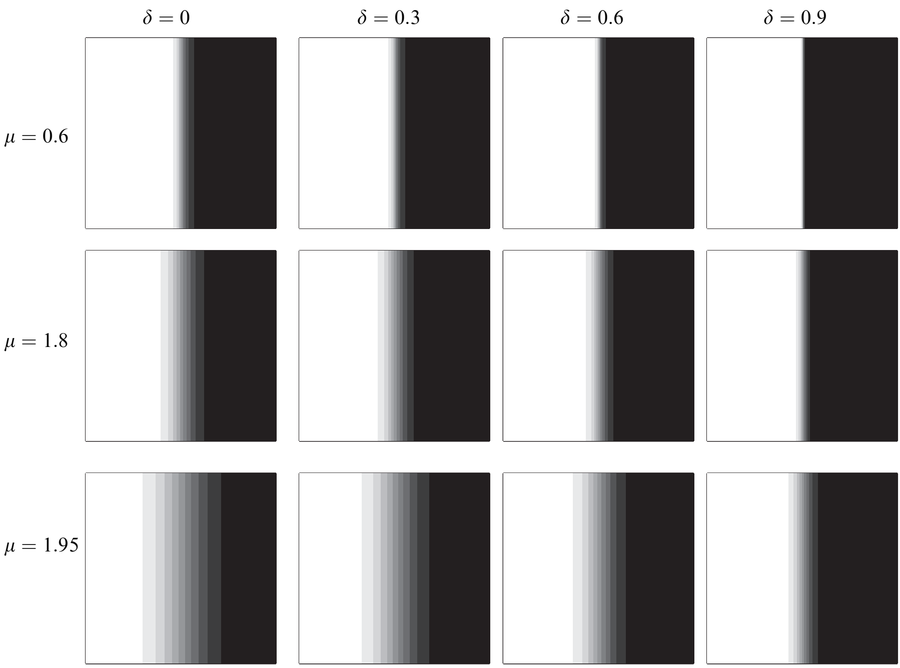

Figure 6. Grey-scale maps showing stationary front solutions of the symmetric model [equations (14)–(16)] in the $x-y$ plane, for different values of μ and δ. The front solutions represent transition zones between the mixed population states N_+ (black) and N_- (white). Darker regions represent higher densities of v-residents. Increasing the parameter δ (mobility associated with status considerations) for a given μ (status gap) leads to a narrowing of the front width (transition zone). The same effect is obtained by decreasing μ (increasing the status gap). Parameters: $\varepsilon = 1$, $\xi = 1$, $\delta_4 = 0.1$, $\sigma_1 = \sigma_2 = \rho_1 = \rho_2 = 0$.

motion is a result of a competition for housing at the interface between the two population states (Downs, 1981; Lynn and McGeary, 1990, page 48; Schwab and Marsh, 1980). The competition in the model equations is affected by the population growth rates, the housing capacities, the nonlocal migration, etc. The manners by which the interface velocity is affected by these factors have been studied by Yizhaq and Meron (2002).

Interface motion may lead to merging of neighbourhoods, as figure 7 demonstrates. The top-left frame in this figure shows the initial condition; a sharp interface separating a v-population (white domain) from a u-population (black domain), with an enclave of u-population inside the domain of v-population (the black square). The parameters chosen are such that the pure population states P_- and P_+ are unstable, but the mixed population states N_- and N_+ are stable. The successive frames (from left to right) show the time evolution of the initial population distribution. The white and black domains first converge to the stable N_- and N_+ states, respectively, and the initial sharp interface widens to its characteristic width. Along with these processes the black domains (N_+) invade the white domain (N_-) and merge into a single black domain that continues invading the white domain.

Invasion processes are generally not uniform; the invasion in one part of the interface may be faster than in another part because of nonuniform conditions, such a spatially varying infrastructure. The model predicts the possibility of nonuniform invasion even when all conditions are uniform. Such an invasion form is a result of an interface instability (Cross and Hohenberg, 1993). Figure 8 demonstrates

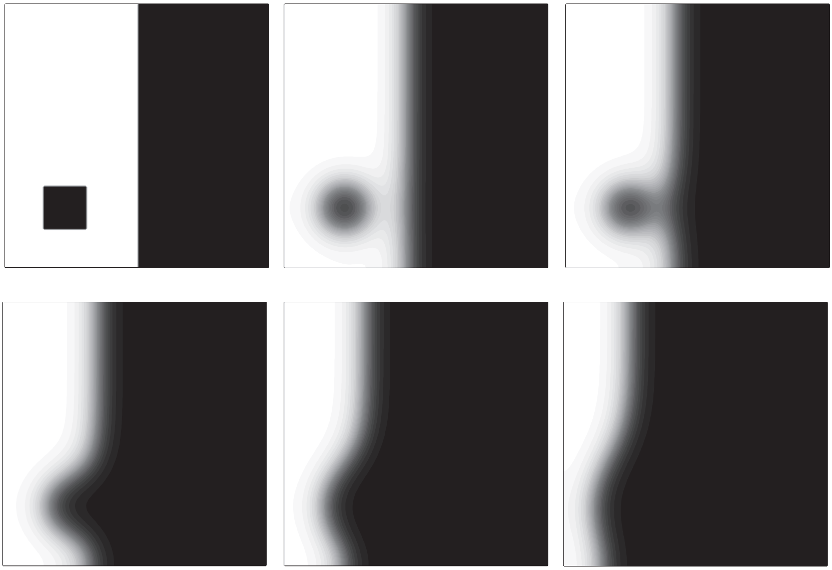

Figure 7. Grey-scale maps showing solutions of the asymmetric model [equations (11) – (13)] in the $x - y$ plane that represent neighbourhood-change processes. The temporal evolution of neighbourhoods (time goes from left to right) shows a hypothetical initial state (top-left frame) involving neighbourhoods of solely v-residents (black) solely u-residents (white) with sharp interfaces (zero width), the convergence to mixed population states N_- (black) and N_+ (white) with wider interfaces, the invasion of the N_- state into the N_+ state, and the merging of the N_- domains. Parameters: $\mu = 1.9$, $\varepsilon = 1$, $\xi = 1$, $\alpha = 1.006$, $\beta = 1$, $\delta_1 = 0.95$, $\delta_2 = 0.6$, $\delta_4 = 0.001$, $\sigma_1 = \sigma_2 = \rho_1 = \rho_2 = 0$.

(a) (b) (c)

Figure 8. Grey-scale maps showing solutions of the symmetric model [equations (14) – (16)] in the $x - y$ plane that demonstrate nonuniform invasion due to a modulational interface instability (time goes from left to right). An initial perturbation along a flat interface (left frame) evolves into a growing bulge of the N_- state. The instability sets in when the parameter δ, which quantifies migration due to status considerations, exceeds a threshold value, δ_c (below δ_c the initial perturbation dies out and the interface resumes a flat form. Parameters: $\mu = 1.88$, $\varepsilon = 1$, $\xi = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta_2 = 1$, $\delta = 1.12$, $\delta_4 = 0.1$, $\sigma_1 = \sigma_2 = \rho_1 = \rho_2 = 0$.

this instability. Starting with a flat interface separating an N_- domain (black region) from an N_+ domain (white region), a small invasion of v-residents into the N_+ domain grows into a bulge which further intrudes into the N_+ domain. The instability sets in when the parameter δ, which quantifies migration due to status considerations, exceeds a threshold value, δ_c. Below δ_c initial perturbations along the interface smooth out and disappear.

The mechanism of nonuniform invasion can be understood by considering the response of u-residents to a local invasion of v-residents to form a bulge in the interface. Perceiving the threat of status decline u-residents respond by migrating out of the bulge. Migration of u-residents to the sides of the bulge inhibits invasion of v-residents to these areas and favours further invasion ahead.

The possible development of modulated interfaces (transition zones) in homogeneous systems has been addressed previously by Rosser (1980) in an attempt to explain wedge-shaped ghettos, and by Downs (1981) who emphasized the importance of economic considerations.

5.3.2 *Tipping process*

The 'tipping-point' phenomenon refers to a sudden reversal of a population distribution as a critical minority population exceeds some threshold. Evidence of urban tipping has been available since the pioneering studies of Grodzins (1957) and Wolf (1963). However, researchers are not in full agreement about the nature of and the driving force behind this phenomenon (Goring, 1978; Hartshorn, 1992; Schwab and Marsh, 1980; Schwirian, 1983; Woods, 1981). In this section we will demonstrate the existence of tipping in the proposed model, and use concepts of dynamical systems theory to gain additional insight into this interesting phenomenon.

Let us consider, for example, a parameter range where the two uniform mixed population states N_+ and N_- exist and are stable. In addition to these stable states the system also has an unstable uniform state: the M state in the symmetric model or the unstable branch of the saddle-node bifurcation in the asymmetric model (see figure 1) which we also denote here by M. To account for the tipping-point phenomenon in the model equations we first restrict the analysis to uniform states. A calculation of the eigenvalues associated with the unstable state M in a wide parameter range shows that two of the three eigenvalues have negative real parts. This implies a two-dimensional stable manifold of the M state which divides the three-dimensional *phase space*, that is, the space spanned by u, v, s, into two distinct parts. The asymptotic behaviour of the system depends on the location (in phase space) of the initial state with respect to the stable manifold of M. If the initial state is located on the same side of the stable manifold as the N_+ state, the system will converge in time to the N_+ state. If on the other hand the initial state is located on the other side of the stable manifold, the time evolution will take the system to the N_- state. This threshold behaviour can be regarded as *tipping*: starting with an N_+ state and perturbing it strongly enough, for example, by increasing v, so as to cross the stable manifold of M, leads to population inversion and convergence toward the N_- state.

A mathematical investigation of phase-space trajectories near the M state suggests four kinds of transient behaviour as figure 9 shows. The transients are affected by two main factors, (1) the direction of the stable manifold at the transient point in phase space, and (2) the proximity to the stable manifold. Consider first the case where tipping fails, that is, over a long time the system converges to the N_+ state. Starting at any point along the A trajectory in figure 9, or at any point along the D trajectory the system will eventually evolve to the same state, N_+. The two trajectories, however, represent different transients, as figure 10 shows, because they correspond to different directions of the stable manifold. In A the v-population first sharply decreases whereas in D it first sharply increases (stage II). Following this fast initial stage is a slow evolution stage where the v-population in both cases decreases (stage III). The difference in the time scale follows from the proximity to the stable manifold. The closer the initial point to the stable manifold, the closer the system approaches the unstable M state, and the sharper the difference in time scales. The next stage (stage IV) is faster as the system departs from the unstable M state, but as it approaches the asymptotic state, N_+, the time evolution is slow again (stage V). Stage I corresponds in all cases to the perturbation (for example, invasion) that posits the system in its initial state.

Consider now the case where tipping succeeds, that is, over a long time the system converges to the N_- state. Starting at any point along the B trajectory, or at any point

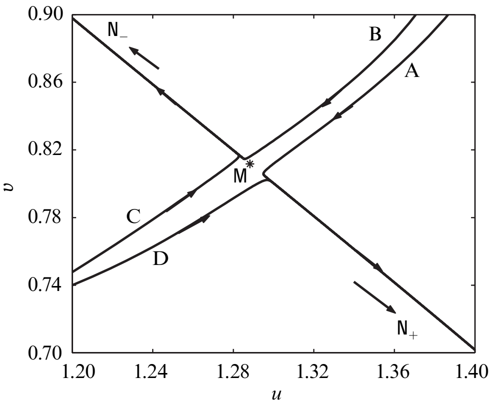

Figure 9. Trajectories in the (u–v) phase plane showing the time evolution of four different initial conditions near the unstable mixed state M. The initial conditions may represent deviations from the stable N_+ state as a result of invasions of v-residents. Trajectories B and C describe invasions that shift the system to the N_- state (tipping) whereas trajectories A and D describe weaker invasions that do not have any effect on the long-run behaviour of the system. The trajectories were obtained using the uniform asymmetric model [equations (11)–(13)]. Parameters: $\mu = 1.57$, $\varepsilon = 1$, $\xi = 1$, $\alpha = 1.1$, $\beta = 1$, $\gamma = 1$.

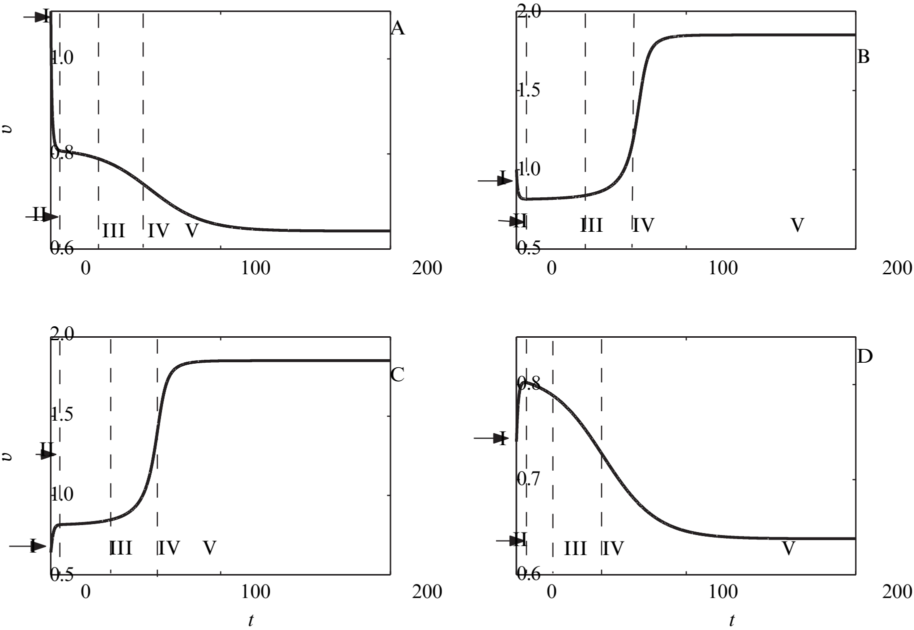

Figure 10. The time dependence of the v population for each trajectory shown in figure 8. Graphs B and C show tipping whereas in graphs A and D tipping fails. In all graphs five stages can be identified: (I) initial condition, (II) transient, (III) slow growth in cases B and C (decline in A and D), (IV) rapid growth (decline), (V) convergence. For more details see the text.

along the C trajectory the system will eventually evolve toward the same state, N_-. The different initial states (pertaining to different directions of the stable manifold of M) lead to different responses in stage II: in trajectory C the υ-population continues the perturbation trend and keeps increasing, while in trajectory B the υ-population first decreases. The subsequent stages are similar in both cases: slow (stage III), fast (stage IV) and slow (stage V) increase of the υ-population toward its asymptotic value at the N_- state.

The discussion of the tipping-point phenomenon has so far been restricted to uniform solutions of the model equations. In this context the initial invasion that induces the tipping was assumed to be uniform in space. Very often the invasion process occurs locally. A 'successful' tipping results in a nucleus of the inversed population that grows or approaches a fixed size. Numerical studies of the nonsymmetric model indicate, however, that the growth of the nucleus is not guaranteed even if population inversion takes place locally; the nucleus must have a minimum size or a critical radius, otherwise it will diminish and disappear. This finding is demonstrated in figure 11. The numerical studies also indicate that the critical radius increases as the status gap increases (or μ decreases).

Why should a minority enclave reach a certain critical size in order to expand into a majority-populated area? A possible answer to this question is as follows: if the relationship between the majority and minority is that of repulsion, the concentration of minority families in the fringe of the majority populated areas should be sufficiently large in order for the majority households to realise that their way of life in the area is threatened and that they may be better off moving elsewhere. On the minority side, reaching a certain level of concentration may trigger 'economies of scale' and the establishment of more specialized ethnic stores and institutions (Massey, 1985).

However, if the critical size is not reached, the nucleus of an invading population may disappear because of exogenous factors, such as immigration and urban policies (slum clearance and gentrification), or because of endogenous causes, such as violent reaction of the local population to 'intruders' (Sampson et al, 1997).

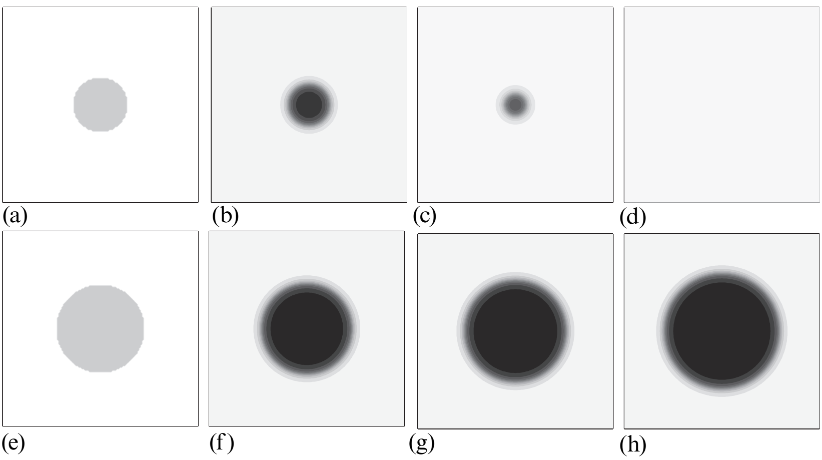

Figure 11. Grey-scale maps showing solutions of the asymmetric model [equations (11)–(13)] in the $x-y$ plane that represent failure [frames (a)–(d)] and success [frames (e)–(h)] of local tipping. Two nuclei of invaded domains [grey disks in frames (a) and (e)], first go through local tipping [black disks in frames (b) and (f)], and then, depending on their initial sizes, either shrink and disappear [frames (c) and (d)], or grow [frames (g) and (h)]. Darker areas represent regions with higher densities of υ-residents. Parameters: $\mu = 1.8$, $\xi = 1$, $\varepsilon = 1$, $\alpha = 1.007$, $\beta = 1$, $\gamma = 1$, $\delta_2 = 1$, $\delta_1 = \delta_3 = 0.8$, $\delta_4 = 0.001$, $\sigma_1 = \sigma_2 = \rho_1 = \rho_2 = 0$.

6 Discussion

The proposed mathematical model can be regarded as an extension of earlier continuum models of intraurban segregation (O'Neill, 1981; Zhang, 1989). The main new ingredient of the proposed model is a status variable. Mediating the interaction between the two population groups by status considerations leads to a wider range of segregation and neighbourhood-change forms compared with earlier models. In particular, we find that segregation patterns may not necessarily involve enclaves of pure populations. At moderate status gaps the model produces weaker forms of segregation patterns involving domains of mixed populations with different proportions, as observed in practice (Morrill, 1995). The different segregation forms, and their dependences on the interaction types between the two populations, are demonstrated in graphs of the dissimilarity index versus the status-gap parameter (figure 3).

We also find that neighbourhood-change processes may not necessarily involve local migration directed from highly populated places to places of lower population density. As the model shows, there are circumstances where status considerations dominate and local migration occurs in the opposite direction, that is, from areas of lower densities to areas of higher densities. In urban systems such migration may be commonplace, reflecting the alternative waves of suburbanization and reurbanization (see, among others, Ishikawa and Fielding, 1998).

The continuum and the deterministic nature of the model help to elucidate dynamical features of segregation phenomena that are less apparent in stochastic discrete models. The relation of segregation to an instability of a mixed population state, M, has not been realized in studies of discrete models, although these models may capture the instability. Likewise, the coexistence of two stable mixed population states, N_+ and N_-, beyond the instability cannot be easily inferred from studies of discrete models. The coexistence of stable states implies two different equilibrium population ratios (u-rich and v-rich) which dictate the local population dynamics; an initial population mix will evolve in time toward one of the two equilibrium population ratios. Another example, where continuum models appear advantageous over discrete ones, concerns the contour shape of a transition zone in the two-dimensional urban plane. Using a continuum model we could identify an instability of the transition zone to undulating perturbations which lead to uneven invasion of one population state into another. Uneven enclave shapes are usually explained by anisotropy of the urban space (Rosser, 1980), but from the point of view of pattern-formation theory, this phenomenon can also occur in a homogeneous and isotropic environment. Studies of the model in two space dimensions can reveal the conditions under which corrugated neighbourhoods develop.

In extending the basic model to include direct interaction terms we assumed a symmetry between the two populations (a single coefficient ζ multiplies both terms). The symmetry and the positive sign of ζ imply repulsive interactions between the two population groups. The nature of the interactions between the two population groups, however, may not be symmetric; although one population group may repel the other, the latter group may be indifferent or even attract the former (Morrill, 1995). Such a situation can readily be simulated with the model by assuming competition coefficients with opposite signs.

An underlying assumption of the proposed model is that the urban area under consideration is homogeneous in respect of infrastructure development, types of housing, accessibility to the city centre, etc. Strictly speaking, this assumption can be justified only for fairly small areas at the spatial scale of a few residential blocks. At larger scales heterogeneities of these types can be handled by assuming spatial variations of the model parameters, or by adding a new dynamical variable reflecting differences in intraurban development. We postpone the consideration of spatial heterogeneities to a future study.

Another topic that deserves special attention is the effect of nonlocal migration. The model equations (8) – (10) include terms describing migration to distant locations, but we have not studied their effect in great detail. Analysis of the uniform states and their stability in the limit of large systems shows that these terms may unfold the bifurcations and shift their locations. The effects of these terms are expected to be more pronounced for nonuniform states. We intend to investigate these effects in a future study.

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